Keeping the next death batch fair

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A fair allocation of today’s deaths can leave the surviving pool unable to allocate the next batch fairly. This paper formulates and tests allocations that preserve feasibility for specified subsequent batch sizes, using the calendar-batch probability law in both intervals.

The distinction changes the answer. In one example, every successor of an ordinary FTP allocation can handle a single death, but almost half cannot handle a pair. In another, every successor can handle singles and pairs, but more than two fifths cannot handle triples. A constrained version of the same entropy objective repairs both examples. An unequal-mortality case gives the same result.

A proposed allocation can be tested against its next operating requirement before settlement. Some states admit a suitable allocation; others cannot satisfy that requirement through reallocation alone. Those cases require a distribution, admission or terminal policy specified in advance.

The experiment being tested

Each account starts with a positive balance. Deaths during the current interval are settled together. Survivors retain their own balances and receive nonnegative shares of the deceased estates. There are no payouts, new members, investment returns or mortality revisions in the examples. Total capital remains 100 arbitrary units.

Fairness is defined separately for each death count: conditional on that count, each member’s expected mortality credit equals their expected forfeiture. This is stronger than fairness averaged across counts. Under independent interval mortality, death-set probability is proportional to the product of the decedents’ mortality odds. Both intervals use those odds, rather than the hazards for chronological next-death processing.

The examples specify relative odds and covered counts, rather than a fitted calendar duration. Multiplying all odds by the same factor leaves each conditional identity law unchanged but changes the probabilities of the counts. Consequently, the proportions of failing successor states below are conditional results. They are not estimated annual failure rates for a pension membership.

Why protecting the next single death is insufficient

Six accounts hold 26, 24, 17, 13, 11 and 9 units. Mortality odds are equal and exactly two members die in the current interval. There are 15 possible deceased pairs, each equally likely.

Ordinary FTP is fair for that current pair law. Consider the event in which the accounts holding 24 and 17 die. The following table shows how the surviving accounts develop. Values are rounded to two decimal places.

Survivor's original balance	After ordinary FTP	After constrained FTP
26	47.50	39.99
13	20.87	24.13
11	17.47	19.86
9	14.16	16.02

The ordinary result passes the familiar single-death condition: nobody holds more than half the fund. But four equal-risk survivors facing a pair of deaths have a stricter individual limit of 40%. A member above that level cannot receive enough expected estate from events in which they survive. The 47.50 account therefore makes the next pair allocation impossible.

Across all 15 current pairs, ordinary FTP produces seven successors that fail the next-pair conditions. Every successor nevertheless passes the next-single-death conditions. The constrained calculation preserves both requirements in all 15 successors. It checks groups of members as well as individual accounts; the 40% cap alone is not the full batch test.

The same mechanism extends to triples. Eight accounts hold 20, 19, 17, 13, 11, 9, 7 and 4 units. Exactly three die now, giving 56 equally likely current death sets. Ordinary FTP leaves every successor capable of a fair single- or pair-death allocation, but 23 successors cannot allocate the next triple fairly. Adding the triple constraints repairs all 56.

The three tested cases are summarised below. The failing-state count refers to ordinary FTP; the final column refers to the constrained policy.

Current pool and deaths	Covered next counts	Failing successors before / after
Six accounts; two deaths; equal odds	One and two	7 of 15 / 0 of 15
Six accounts; two deaths; unequal odds	One and two	2 of 15 / 0 of 15
Eight accounts; three deaths; equal odds	One, two and three	23 of 56 / 0 of 56

The unequal-odds case uses balances (25,22,17,14,12,10) and relative odds (0.9,1.1,1.05,0.85,1.2,1). Its two failing current events together have approximately 13.10% probability under the conditional pair law. Their probability is not simply two fifteenths, because the events are not equally likely.

What the algorithm changes

The objective is an expected-capital-weighted forward-KL correction. Current fairness, estate conservation and survivor eligibility are constraints. For every possible current death set, further constraints require the surviving balances to admit a fair allocation in each covered next-count layer.

Those added conditions are linear in the current credits when the next mortality law is fixed independently of the allocation. The optimisation therefore remains convex. If a feasible policy exists, strict convexity selects a unique current transfer table. If ordinary FTP already meets the additional requirements, the optimum remains unchanged.

There is a qualification to the nominal-gain interpretation. Under these fixed margins, every positive member-only proportional reference has the same forward-KL minimiser. The continuation constraints do not alter that cancellation. Nominal gain remains one explanation of the reference, but the calculation does not establish that it uniquely supplies the economic objective. Nor does entropy optimality imply minimum pension-income variance or maximum member welfare.

The examples request a small positive margin: each relevant future Hall slack must be at least one ten-thousandth of that next layer's expected deceased estate. This is a numerical design margin, not a probability of failure. The optimiser's proposed policies are checked independently against the actual zero-feasibility boundary.

Allocation changes are measurable. Averaging over current events, half the sum of absolute credit changes is approximately 2.98 units in the equal-pair case, 0.66 in the unequal-pair case and 3.01 in the triple case. This measures capital directed to different survivors in different outcomes. It is neither a fee nor a change in aggregate capital or expected member wealth under exact fairness.

Verification and computational scope

The optimiser enumerates current death sets and future member subsets. The pair examples have 60 permitted credit variables and 210 future constraints. The triple example has 280 variables and 2,520 future constraints. Recorded numerical solves took approximately 0.05 to 0.40 seconds. These are small research examples, not evidence of large-pool performance.

A separate audit reconstructs the conditional laws directly and evaluates the fairness equations and all future Hall conditions using rational arithmetic. It does not import the optimiser's probability calculations, constraint matrix or stopping criteria. Its first pass treats the binary64 candidate entries as exact numbers and normalises every column to conserve its estate. Maximum current fairness error across the three constrained policies is below 3e-13 units; every covered successor is strictly feasible.

We then round the candidate credits to rational numbers and correct the remaining expected-flow margins along a spanning tree of permitted transfers. The resulting three policies satisfy current fairness and estate conservation exactly. The independent audit accepts them with zero fairness tolerance and verifies strictly positive Hall gaps for every covered successor and next count. No credit changes by as much as 5e-10 units from the numerical proposal. These are exactly fair feasible policies close to the numerical optima; the correction does not supply an exact proof of entropy optimality.

The audit itself has been checked against an independently implemented exact transportation maximum-flow calculation, including feasible and infeasible pools. The checks distinguish a zero-margin feasible boundary from impossibility and include the Hall inequalities for groups larger than the death count when reporting the minimum margin. Separate exact tests establish the probability and cut formulas. None of this certifies mortality estimates, death records or subsequent rounding to pennies.

Compressed ordinary-batch calculation remains relevant when its allocation passes continuation checks. The constrained prototype stores the event table explicitly. A scalable method for imposing future constraints remains an implementation question; the short running times in these small examples do not settle it.

Some pools cannot be repaired by reallocating credits

The successful six-member example starts its largest account at 26. A nearby example has balances (27,23,17,13,11,9), retaining total capital of 100 and the same current and next pair laws. No fair current allocation can leave every successor capable of a fair next pair.

The proof is short. The account survives the current pair with probability two thirds. In every surviving state, next-pair feasibility would limit it to 40. Its expected closing account could therefore be no more than 80/3, or approximately 26.67. Current fairness requires that expectation to equal its starting balance of 27. The requirements contradict one another.

This is an economic obstruction, not a solver-convergence problem. It supplies a check before optimisation. It also shows why the policy question cannot be deferred until the software reports a failure: a pool may admit a fair current allocation even though no such allocation can preserve the chosen next-period requirement.

Demanding separate fairness for every possible next count is more restrictive still. With m surviving members and positive independent mortality odds r_i, feasibility for all counts from one to m-1 requires, and is implied by, the condition that balance times mortality odds is equal across members. The count leaving only one survivor establishes necessity. Equal estate shares among survivors establish sufficiency for each count.

That is a statement about the current state. The equal-share allocation need not preserve the same condition afterwards. It also excludes the all-dead event, for which no survivor exists. Neither a short interval nor a very small probability removes these states from a model that promises fairness separately for every positive-probability count.

What this adds to the pension design

The results provide a test of whether a proposed allocation leaves the next covered batch feasible, a constrained calculation repairing three explicit failures, and a proof that a nearby failure cannot be repaired. These distinctions can guide an operating rule: use ordinary FTP where admissible, seek an alternative where possible, and identify states requiring a different transition or settlement policy.

The open choices concern the member-specific distributions, admission and mortality-review dates, covered counts and the treatment of uncovered or terminal events. Excluding a count from an optimisation is not a settlement policy for that count. The present calculations also cover only the next allocation: they do not prove that the policy chosen then can preserve every later successor.

The next design comparison should put specified ladder distributions into these transitions and measure which states remain admissible. Common proportional distributions will not change relative feasibility; member-specific distributions can. Their amounts must follow an agreed entitlement rule, rather than being changed after a death to rescue an inconvenient calculation.

The literature includes periodic nominal-gain allocation, multiple-death adaptations and dynamic fairness. Some exact alternatives permit payments to deceased estates, changing recipient eligibility. This paper states a continuation formulation, tested repairs and checkable certificates for survivor-only settlement. The calculations do not establish academic priority.

Technical appendix

A. Calendar identity laws and future Hall cuts

For independent interval death probabilities q_i, let $r_i = q_i/(1-q_i)$. Conditional on k deaths, a deceased set D has probability

$$\pi_D = \frac{\prod_{j \in D} r_j}{e_k(r)}, \qquad |D| = k.$$

Here e_k is the elementary symmetric polynomial. Let C_{iD} be a survivor credit and s_i the opening balance. Current fairness requires the expected credit to equal s_i times the conditional death probability, and each column of C must sum to the deceased estate. Before any specified retention, the successor balance is $z_i = s_i + C_{iD}$.

On a fixed successor membership, let F be the next death set, α_i its member death probability and p_U the probability that every member of U dies. A future fair allocation exists precisely when every proper nonempty subset U satisfies

$$p_U \sum_{i \in U} z_i \leq \sum_{j \notin U} (\alpha_j - p_{U \cup \{j\}}) z_j.$$

This compares U's inaccessible estate with the expected forfeiture available outside U. It is the weighted transportation Hall condition rearranged into fixed coefficients multiplying successor balances. For the next count l,

$$p_U = \frac{(\prod_{i \in U} r_i) e_{l-|U|}(r_{-U})}{e_l(r)}.$$

The probability is zero when |U| exceeds l. Such proper subsets have a positive Hall slack equal to their omitted targets. Future odds may differ from current odds, but they must be supplied independently of the allocation for this linear formulation. Known positive retention factors multiply the corresponding balances and credits. A full payout requires its own terminal or membership transition.

B. The hierarchy of batch checks

For a specified future count, checking individual accounts alone is insufficient. With five equal-odds accounts (41,41,6,6,6) and two deaths, every singleton cut passes but the largest pair has a Hall deficit of exactly 1. With seven equal-odds accounts (32,32,32,1,1,1,1) and three deaths, every singleton and pair cut passes but the largest triple has a deficit of 36/35. These are tests within a fixed future count, distinct from comparing future counts one, two and three in the main examples.

For a future pair layer, the complete singleton and pair cuts can be evaluated in quadratic arithmetic in successor membership after linear moment calculations. That reduces the cost for one known successor state; it does not eliminate the combinatorial number of possible current death sets.

C. The all-count restriction

Let S be total survivor capital and consider the next count m-1. Only member i survives with probability

$$t_i = \frac{1/r_i}{\sum_j 1/r_j}.$$

Its closing account is then S, otherwise zero. Fairness requires z_i=S t_i, so z_i r_i must be constant. Conversely, that condition makes equal division of each deceased estate among its m-l survivors fair for every fixed count l<m. The theory note proves the elementary-symmetric-polynomial identity used for this converse and checks it with exact fractions. This establishes one-state feasibility, without asserting dynamic invariance.

D. Verification and related literature

The verification uses numerical candidate tables, rational policies and separate checks of fairness, conservation and successor feasibility. The rational certificate establishes feasibility and fairness; it does not prove that the corrected rational table is the entropy optimum.

Fullmer and Sabin's *Individual Tontine Accounts* describes the periodic nominal-gain rule and its finite-pool fairness approximation. Publisher manuscript.

Hieber and Lucas's *Modern Life-Care Tontines* proves a dynamic fairness result with mortality credits also payable as death benefits. Institutional manuscript.

Zhou and co-authors' published 2025 ASTIN paper distinguishes an approximate survivor-only rule from an exact rule that permits deceased recipients. These eligibility conventions require separate comparisons. Publisher full text.

Dhaene and Milevsky price an administrator's role in the all-dead event, providing a comparator for terminal treatment. This does not extend survivor-only eligibility. Author manuscript. Weinert and Gründl's multiple-death adaptation provides a further comparison whose event and recipient conventions must be distinguished.

Russ Oxley directed the research and its economic interpretation. AI systems contributed to derivations, calculations, code and drafting. Each paper states the assumptions and checks supporting its conclusions. Responsibility for the published work rests with the author.