Keeping a tontine fair as the pool changes

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A fair tontine transfer leaves each member's expected wealth unchanged by mortality pooling. The deceased lose their accounts; survivors receive the forfeitures. Fairness requires those expected gains and losses to balance for each member, under the specified mortality assumptions.

Fair transfers change survivors’ balances and therefore the feasibility of later allocations. This paper asks when a transfer that is fair under the current mortality model preserves the ability to allocate the next deceased estate fairly.

An allocation can be exactly fair today yet make fair continuation impossible. An exact example below shows that ordinary proportional-weight FTP can do this even when another fair allocation preserves the next step. Adding linear continuation constraints retains a convex optimisation problem, although the resulting rule generally requires more than one fixed set of member weights.

The examples use small artificial pools to expose the mechanism; they do not estimate its frequency in a large pension pool. Proofs and exact-arithmetic examples are included. The connection from period fairness to lifetime value is established probability and appears in the tontine literature. No claim of academic priority is made.

What the existing rule establishes

For any possible death set, survivors retain their accounts and divide the deceased estate. The allocation rule gives each survivor a share proportional to a calibrated member weight. The same weights govern every possible death set of the specified size. They are calibrated before using the identities of the actual decedents.

Fairness fixes each member’s expected incoming credit without usually determining every payment in every outcome. The entropy criterion selects one allocation from the remaining choices. This allocation can also be described as the closest fair correction of nominal gain, using an expected-capital-weighted forward KL distance.

There is a qualification: any strictly positive, member-only proportional reference gives the same optimum under those margins. The nominal-gain reference does not uniquely select it. Nor does this distance measure the effect of today's allocation on tomorrow's feasible choices. That requires an additional condition.

Fullmer and Sabin distinguish exact sequential methods from their simpler periodic nominal-gain rule, whose finite-pool bias they acknowledge. Our work addresses exact fairness for a declared event law; it should not be described as the first treatment of periodic mortality pooling. [1]

A fair transfer can exhaust the ability to continue fairly

Consider three members, Alice, Ben and Cara, with accounts of £40, £30 and £30. Assume independent, identical exponential remaining lifetimes: each current member is equally likely to die next, and this remains true of the survivors. Ignore investment returns, payouts and new members.

These first-death credits follow the proportional-weight rule with weights 2:1:1:

First death	Credit to Alice	Credit to Ben	Credit to Cara	Accounts afterwards
Alice	£0	£20	£20	£0, £50, £50
Ben	£20	£0	£10	£60, £0, £40
Cara	£20	£10	£0	£60, £40, £0

Every row distributes the deceased account exactly. Alice's expected credit is £40/3, matching her expected forfeiture of £40/3. Ben and Cara each expect £10 of credits and £10 of forfeiture. The first transfer is fair.

If Ben dies, Alice and Cara then own £60 and £40. The next survivor must receive the entire £100. Each has an equal chance of being that survivor, so each has expected final wealth of £50. Alice therefore faces an expected loss of £10; Cara an expected gain of £10. No recalibration can change this because there is only one eligible recipient in either outcome.

The obstruction occurs after a successful fair allocation. It does not require a simultaneous-death event, an inaccurate numerical solver or an incorrect mortality model.

Sometimes a different fair allocation preserves the next step

The previous pool cannot run off fairly from its unequal starting accounts. A more informative test asks whether the selection rule itself can discard a feasible continuation.

Take four members with balances £28, £24, £16 and £12, totalling £80. Retain the same equal-mortality assumptions. A fair allocation exists for the next death because no member holds more than half the capital.

The ordinary positive proportional-weight FTP allocation necessarily gives Alice more than £12 when Ben dies. Her balance then exceeds £40. In an equal-mortality pool, an account above half the capital cannot be treated fairly at the next death: its expected forfeiture exceeds all the credits it can receive. The short algebra establishing this result is in the appendix.

Now consider a different table of credits:

First death	Alice receives	Ben receives	Cara receives	Dan receives
Alice: £28	£0	£15	£3	£10
Ben: £24	£11	£0	£12	£1
Cara: £16	£11	£4	£0	£1
Dan: £12	£6	£5	£1	£0

Each row distributes exactly the account shown. Alice's credits across the other three possible deaths sum to £28; Ben's to £24; Cara's to £16; Dan's to £12. Since each death has probability one quarter, expected credits equal expected forfeitures for every member.

The largest surviving accounts after the four possible deaths are £39, £39, £39 and £34. All possible successor pools remain strictly below the £40 boundary. Each therefore admits another fair single-death allocation. This table protects one further death; it does not promise fairness through complete run-off.

This is a selection problem with an exact solution. Both allocations satisfy current fairness. Only the second preserves feasibility at every immediately following state. Its donor-recipient payments are asymmetric, so it cannot be represented by the ordinary common-weight FTP rule under equal mortality.

The table is a feasible witness, not a claim to have found the entropy optimum subject to continuation constraints. It shows that those constraints can change the answer without requiring an external subsidy or a debit to a survivor.

The algorithmic extension

The current optimisation asks for a fair distribution of each deceased estate and then selects the entropy optimum. We can add a further requirement: after each supported death outcome, the resulting balances must admit the next promised allocation.

The existing Hall conditions answer whether a fair allocation exists. When the next mortality law is fixed independently of the transfer amounts, these conditions are linear in the next balances. Next balances are current balances plus the chosen credits. Consequently, the continuation conditions are linear in the current decision variables too.

This gives a defined research algorithm: choose the closest fair allocation subject to specified future-feasibility constraints. The entropy objective remains convex after adding them. If the feasible set is nonempty, the allocation is unique under the same strictly convex objective. However, the new constraints distinguish death outcomes, so their optimality conditions generally require event-dependent adjustments rather than a single vector of member weights.

The advantage is substantive: we have an example in which this broader class preserves a fair next step and ordinary FTP does not. The cost is computational.

Listing every current death set and every future Hall subset is too large for a practical pool. The next computational question is whether conservative continuation bounds or a method that adds only violated constraints can preserve enough of the earlier polynomial compression. This note establishes the formulation and an exact witness; it does not supply a scalable implementation of the extended optimiser.

A finite continuation horizon must be specified. Protecting the next death, protecting the next month and protecting a target-date distribution are different requirements. So are requiring every modelled outcome to remain feasible and allowing a stated response to rare outcomes.

What a promise to the last survivor would require

A closed pool provides a sharp boundary case. Suppose capital stays fixed, nobody joins, there are no payouts or charges, deaths occur sequentially, and the final survivor receives all the capital. For fairness to hold at every history, each current account must equal:

$$\text{pool capital} \times \text{conditional probability of being the final survivor.}$$

The reasoning is short. Repeated fair transfers preserve conditional expected wealth. At the end, a member either receives the entire pool or receives zero. The expected value of that terminal outcome must therefore equal their account at every earlier stage.

With identical independent exponential lifetimes, all current members have the same chance of being last. Their accounts must be equal if exact fairness is promised throughout run-off. The £40/£30/£30 example fails this necessary starting condition.

Heterogeneous mortality does not itself prevent fair run-off. With independent exponential death intensities in the ratio 1:2:3, the probabilities of being last are 35/60, 16/60 and 9/60. Starting balances of £35, £16 and £9 admit a fully fair run-off. These are deliberately matched balances, not a proposed admission formula for ordinary pension saving.

There is a further limit to the entropy selector. For four different exponential intensities, the unique transfers preserving full run-off can be fair and positive but impossible to express using common survivor weights. The appendix gives an integer example. This establishes a conflict with eventual fair continuation, which need not mean the immediately following death is already infeasible.

The closed-pool theorem does not describe a target-date ladder. Scheduled distributions move value out before the final death; new members change capital and composition. The accounting must include those payments and the remaining account. The terminal example shows why payout and closure rules belong in the mathematical contract. Admission can support feasibility, but suitable future entrants cannot be guaranteed.

Hieber and Lucas already connect current accounts to conditional future benefits by backward induction. Their design also permits credits to deceased estates. That broader eligibility changes the feasibility problem. Weinert and Gründl explicitly model multiple deaths and replacement members. Neither arrangement should be treated as the closed, survivor-only experiment above. [2, 3]

A death batch also contains a timing choice

The final deceased set does not always determine the outcome of sequential FTP. Starting from four equal-mortality accounts of £54, £44, £36 and £26, processing Cara's death and then Dan's leaves Alice and Ben with £96 and £64. Reversing the order leaves £94 and £66. Both paths use fair recalibration at each step, and all first-step successor states are feasible.

For the experiment consisting of the next two deaths under identical exponential lifetimes, the two orders are equally likely conditional on their identities. Averaging them gives £95 and £65 for this pair. Applying the correct order average for all possible pairs produces an exactly fair unordered two-death rule. In this example that rule is not the common-weight entropy rule.

Processing each known pair by member-label order instead creates expected net transfers of £1, £2/3, minus £2/3 and minus £1 across the four members. The underlying single-death formula has not changed; the imposed order has changed the experiment.

Thus the distinction between chronological processing and a batch contract survives exact arithmetic. Uniform order averaging is justified in this equal-mortality example, not in general. Heterogeneous lifetimes require their conditional order probabilities, and a fixed calendar bucket is a different experiment from the next two deaths. The accompanying checks make these distinctions explicit.

Results and limitations

Numerical batch-allocation studies address fairness under specified event laws in tested heterogeneous pools. This paper adds a separate requirement: whether an allocation preserves the ability to meet a stated future fairness condition.

There are three established results within the stated models. A fair allocation can lead to an infeasible successor. A different fair allocation can sometimes preserve that successor, requiring departure from common weights. And, with a fixed next-event law, the relevant continuation conditions can be added as linear constraints to the entropy problem.

The practical research target should therefore be a finite-horizon, survivor-only allocation tied to the ladder's actual payout and entry rules. The closed run-off theorem supplies a boundary test. It should not become a substitute pension design. Our current results do not show how often continuation constraints bind in a large pool, how much they change realised credits, or how quickly the extended problem can be solved.

The literature already contains heterogeneous pools, multiple-death models, dynamic fairness and correlated mortality. [1-5] Comparisons must specify recipient eligibility, event laws, feasibility, selection and verification. The results here make no claim of academic priority; a detailed comparison with the related methods remains necessary.

Technical appendix

A. Fairness through time and numerical error

Let H_t contain the settled history and all information used to fix the membership, balances and mortality law before event t. Let D_t be its deceased set, K_t its count, s_{it} the exposed balance and C_{it} the incoming credit, zero for a decedent. Define the net mortality transfer G_{it} as C_{it} minus s_{it} if the member dies, and C_{it} otherwise.

If E[G_{it} | H_t, K_t=k]=0 for every count with positive conditional probability, averaging over counts gives E[G_{it} | H_t]=0. The cumulative mortality-transfer ledger is then a martingale: its conditional expected next increment is zero. This is the tower property. It requires the actual conditional event law and a valid allocation for every state in its scope; it does not establish their existence. Checking only selected counts does not prove unconditional fairness.

With investments and external flows, subtract investment profit and contributions from the account change, then add withdrawals and charges back, to obtain this mortality ledger. It is not generally the entire investment account that has zero expected change. Members who die must remain in the calculation with their forfeitures recorded.

If the audit instead gives conditional expected-error bounds epsilon_itk covering every possible count, define e_{it} as their average using the pre-event count probabilities. For a finite horizon, or a bounded stopping rule based on observed history,

$$\left|\mathbb{E}\left[\sum_{t\le\tau} G_{it}\right]\right| \le \mathbb{E}\left[\sum_{t\le\tau} e_{it}\right].$$

Subtracting each conditional mean gives a martingale; the triangle inequality gives the bound. It measures accumulated expected bias, not the dispersion of realised member outcomes. Unbounded operation needs additional integrability assumptions. Unsupported events need an actual contractual response; a small tail probability does not define one.

B. The four-member example

Write C_{ij} for the credit to i when j dies. Under equal next-death probabilities, fairness requires each row sum of C to equal s_i; conservation requires each column sum to equal the deceased balance. In the positive common-weight rule, with W the sum of weights,

$$C_{ij} = s_j \tfrac{w_i}{W - w_j} = a w_i w_j \quad (i \ne j)$$

for a positive constant a. To obtain the last equality, substitute the row constraints and rearrange to get s_i=a w_i(W-w_i). Thus C is symmetric. For balances (28,24,16,12), the row sums imply C_12 minus C_34 equals (28+24-16-12)/2=12. Positivity gives C_12>12. When member 2 dies, member 1 therefore holds more than 40.

For equal single-death probabilities, the exact next-event Hall gate is max_i s_i at most S/2. The alternative table in the main text has the same row and column sums and leaves every surviving balance at most 39. It therefore proves strict next-event feasibility in every outcome.

A necessary admission bound for n at least three equal-mortality members, preserving another fair single-death step under all outcomes, is s_i at most (n-1)S/(2n). After survival the account must be at most S/2; survival probability is (n-1)/n; fairness equates its expected next balance to s_i. This singleton bound is not sufficient. With four equal-mortality balances (3,3,1,1), it holds, but each size-3 member needs a credit of 1 from each other death to remain fair within the successor cap of 4. Either size-1 deceased account cannot pay both. Future subset constraints remain necessary.

C. General continuation constraints

After current death set D, let $z_{iD}=s_i+C_{iD}$ for survivors and zero otherwise. Conditional on that state, write F for the next deceased set and alpha_iD for its marginal death probabilities. Assume this next law is specified independently of the chosen credit amounts. Next-step Hall feasibility requires, for every subset U of the remaining members,

$$\sum_{i \in U} \alpha_i^D z_{iD} + \mathbb{E}_D\left[\mathbf{1}_{\{U \subseteq F\}} \sum_{j \in F} z_{jD}\right] \le \sum_j \alpha_j^D z_{jD}.$$

Every coefficient is fixed by the next law. Hence these are linear inequalities in z and in the current credits. Combined with current fairness, nonnegativity and conservation, they define a convex feasible set. Weak inequalities allow boundary solutions with zero credits. A practical positive-weight or interior method needs margins on the nontrivial inequalities.

The earlier cancellation of a separable nominal reference still applies: the current fairness and estate margins remain fixed. The changed allocation comes from the continuation constraints, not from a new claim that nominal gain uniquely determines the objective.

The result presumes the frozen balances are the next exposed balances. Specified payouts, returns or admissions must be included in the intervening state transition. If the transition is affine and the next mortality law is fixed, the same linear argument applies. If decisions alter mortality or nonlinear state changes, this conclusion requires a fresh derivation.

D. Closed run-off and the limits of common weights

Let L be the final survivor and S the fixed total capital. Exact conditional fairness makes account X_i a bounded martingale terminating at S times the indicator that L=i. Thus X_i(H)=S P(L=i | H). This necessity follows from conditional expectation, without any exponential assumption.

For independent exponential remaining lifetimes, let h_i(A) be the probability that i is last among active set A. It is

$$h_i(A) = \int_0^\infty \lambda_i e^{-\lambda_i u} \prod_{j \in A \setminus \{i\}} (1 - e^{-\lambda_j u})\, du.$$

Starting with s_i=S h_i(A), a death of j gives survivor i credit S[h_i(A without j)-h_i(A)]. Removing a competitor cannot reduce the chance of being last. Memorylessness supplies the same next-state law, so these nonnegative credits conserve capital and are fair at every step. For general lifetime models, new information can lower a survivor's conditional terminal value; the resulting martingale may require debits or between-death adjustments that survivor-only FTP does not permit.

For intensities (1,2,3,4), take capital 2,520 and balances (1,389,608,327,196). The unique first-death credits compatible with fair complete run-off are:

Death	Probability	To member 1	To member 2	To member 3	To member 4
1	1/10	0	624	441	324
2	2/10	312	0	168	128
3	3/10	147	112	0	68
4	4/10	81	64	51	0

The ratio of member 1's credit to member 2's is 147/112 after death 3, but 81/64 after death 4. The ratios differ, whereas common weights require them to agree. This positive fair allocation establishes interior feasibility; the entropy optimum is common-weight and therefore differs from the unique run-off-preserving rule. The mismatch proves loss of complete fair continuation in at least one branch. Exact checks cover all 11 active subsets of sizes two to four.

Sources and verification

[1] Fullmer, R. K. and Sabin, M. J. (2019), *Individual Tontine Accounts*, Journal of Accounting and Finance 19(8), pp.31-61; especially pp.33-34. Publisher PDF.

[2] Hieber, P. and Lucas, N. (2022), *Modern Life-Care Tontines*. Read in the September 2021 manuscript, equation (5) and theorem 2.6. Author manuscript.

[3] Weinert, J.-H. and Gruendl, H. (2021), *The Modern Tontine*, European Actuarial Journal 11, pp.49-86, section 4.3. Publisher full text.

[4] Milevsky, M. A. and Salisbury, T. S. (2016), *Equitable Retirement Income Tontines: Mixing Cohorts Without Discriminating*, sections 1 and 6.3. Author manuscript.

[5] Zhou, Y., Garces, L. P., Shen, Y., Sherris, M. and Ziveyi, J. (2024), *Risk-sharing Rules for Mortality Pooling Products with Stochastic and Correlated Mortality Rates*, CEPAR Working Paper 2024/26, sections 2-3. Institutional manuscript.

Sabin’s Fair Tontine Annuity (2010) is a reference for the single-death framework. No academic priority is claimed here.

The derivations and exact examples establish the mathematical distinctions and continuation witness. Reported large-pool timings and certificates for ordinary batch allocation do not establish the scalability of the continuation-constrained optimiser.

Russ Oxley directed the research and its economic interpretation. AI systems contributed to derivations, calculations, code and drafting. Each paper states the assumptions and checks supporting its conclusions. Responsibility for the published work rests with the author.